Ramanujan sums analysis of long-period sequences and $1/f$ noise

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Abstract. Ramanujan sums are exponential sums with exponent defined over the irreducible fractions. Until now, they have been used to provide converging expansions to some arithmetical functions appearing in the context of number theory. In this paper, we provide an application of Ramanujan sum expansions to periodic, quasi-periodic and complex time series, as a vital alternative to the Fourier transform. The Ramanujan-Fourier spectrum of the Dow Jones index over 13 years and of the coronal index of solar activity over 69 years are taken as illustrative examples. Distinct long periods may be discriminated in place of the $1/f^{\alpha}$ spectra of the Fourier transform.

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1. Introduction

Signal processing of complex time-varying series is becoming more and more fashionable in modern science and technology. Indices arising from the stock market, changing global climate, communication networks such as the Internet etc. are widely used tools for business managers or governmental representatives. There already exists a plethora of useful approaches for signal processing of complex data. The oldest and perhaps most widely used method is a Fourier analysis and its "fast" implementation: the fast Fourier transform (or FFT). Other complementary techniques such as wavelet transforms, fractal analysis and autoregressive moving average models (ARIMA) were developed with the aim of identifying useful patterns and statistics in otherwise seemingly random sequences [1].

Ramanujan sums are defined as power sums over primitive roots of unity. One can use an orthogonal property of these sums (closely related to the orthogonal property of trigonometric sums) to form convergent expansions of some arithmetical functions related to prime number theory [2, 3]. Following the ideas of Gadiyar and Padma [4], the first author proposed to expand the domain of application of Ramanujan sum analysis from number theory to arbitrary real time series and introduced the concept of a *Ramanujan-Fourier transform* [5]. This earlier work remained quite ambiguous

about the detection of isolated periods. Ramanujan sum expansions of divisor sums, sums of squares, and the Mangoldt function are well known. Surprisingly, the detection of a singly periodic signal by the Ramanujan sum analysis has not been considered before. But the Ramanujan-Fourier amplitude corresponding to a single cosine function of period q is extremely simple: as we shall see, the amplitude of the cosine function is simply scaled by the cosine of the delay and the inverse of the Euler totient function $\phi(n_0)$. Similarly to the standard discrete Fourier transform, there are spurious signals of magnitude $O(n_0/t)$, depending of the length t of the averaging spectrum.

In the discrete Fourier transform, a sample to be analyzed is discretized into pieces of length $1/q$ and the expansion is performed over the q-th complex dimensional vectors of the orthogonal basis $e_q^{(p)}(n) := \exp\,(\frac{2i\pi p}{q}n)$, $(p = 1, \ldots, q)$. The orthogonal property reads $\sum_p e_q^{(r)}(n)e_q^{(s)}(n) = q\delta(r,s)$, where $\delta(r,s)$ is the Kronecker symbol. The expansion of a time series is $a(n) = \sum_p a_p e_q^{(p)}(n)$ with Fourier coefficients $a_p = \frac{1}{q}\sum_n a(n)e_q^{(p)}(-n)$, where the summation runs from 0 to $q-1$. In the Ramanujan-Fourier transform, the expansion $a(n) = \sum_q a_q c_q(n)$ over the Ramanujan sums $c_q(n) = \sum_p' e_q^{(p)}(n)$ (see Sec. 2) involves the resolution $\frac{1}{q}$ at every single scale from $q = 1$ to $t \to \infty$. The deep principle behind rests on a very intricate link between the properties of irreducible fractions $\frac{p}{q}$ and prime numbers [2]-[5]. As a result, one finds a much finer structure of time series, with a variety of novel features.

The paper is organized as follows. In Sec. 2, we remind the reader with the arithmetical properties of Ramanujan sums, provide the definition of the Ramanujan-Fourier transform and examine the detection of a cosine signal. In Sec. 3, the use of the method is illustrated on the data from the stock market and solar activity.

2. Ramanujan sums and the Ramanujan-Fourier transform

Ramanujan sums are real sums defined as n-th powers of q-th primitive roots of the unity,

$$c_q(n) = \sum_p{}' \exp\,(2i\pi\frac{p}{q}n),$$

where the summation runs through the p's that are coprime to q (hence the use of the symbol "$'$"), being first introduced in the context of number theory [2, 3] for obtaining convergent expansions of some arithmetical functions such as the relative sum of divisors $\sigma(n)/n$ of an integer number n,

$$\sigma_n/n = \sum_{q=1}^{\infty} \frac{\pi^2}{6q^2} c_q(n).$$

They are multiplicative when considered as a function of q for a fixed value of n, which can be used to prove an important relation

$$c_q(n) = \mu(q/q_1)\frac{\phi(q)}{\phi(q/q_1)},\ \ q_1 = (q, n).$$

In the above relation, the Euler totient function $\phi(q)$ is the number of positive integers less than q and coprime to it. The Möbius function, $\mu(n)$, vanishes if q contains a square in its (unique) prime number decomposition $\prod_i q_i^{\alpha_i}$ (q_i a prime number), and is equal to $(-1)^k$ if q is the product of k distinct primes. One can readily checks the following orthogonal property

$$\sum_{n=1}^{rs} c_r(n)c_s(n) = 1 \text{ if } r = s \ \text{ and } \ \sum_{n=1}^{q} c_q^2(n) = q\phi(q) \text{ otherwise.}$$

For an arithmetical function $a(n)$ possessing a Ramanujan-Fourier expansion

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n),$$

with Ramanujan-Fourier coefficients a_q, one can write a Wiener-Khintchine formula, relating the autocorrelation function of $a(n)$ and its Ramanujan-Fourier power spectrum,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N} a(n)a(n+h) = \sum_{q=1}^{\infty} a_q^2 c_q(h).$$

This relation was used for counting the number of prime pairs within a given interval [4]. A similar formula has been proposed for the convolution and cross-correlation [6].

Clearly, the Ramanujan sum analysis of an arithmetical function looks like the Fourier signal processing of a time series $a(n)$ at discrete time intervals n. This formal analogy was developed in [5] for the processing of time series with a rich low frequency spectrum [7]. Ramanujan signal processing was further developed in the context of quantum information theory [8] eventually leading to an original approach of quantum complementarity [9]. The Ramanujan-Fourier transform was also used for processing time series of the shear component of the wind at airports [10], the structure of amino-acid sequences [11] and in relation to the fast Fourier transform [12]. All these applications make use of the property that for arithmetical functions possessing a mean value

$$A_v(x) = \lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{t} a(n),$$

one can write the inversion formula

$$a_q = \frac{1}{\phi(q)} A_v(a(n)c_q(n)).$$

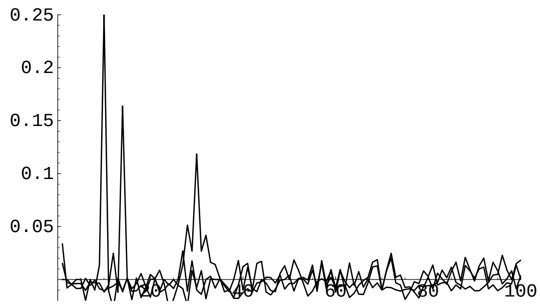

Figure 1. Ramanujan-Fourier spectrum for three different cosine functions of periods $n_0 = 10$, 14 and 30, computed from a sample of length $t = 100$. The amplitude at $q = n_0$ equals $1/\phi(n_0) = 1/4$, 1/6 and 1/8, respectively.

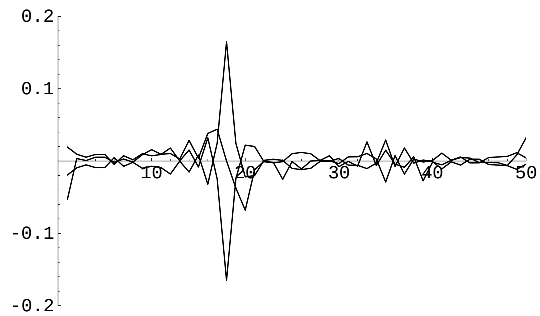

Figure 2. Ramanujan-Fourier spectrum for the cosine function of period $n_0 = 38$, with delays $\delta = 0$, $\pi/2$ and π, and sample length $t = 100$. The amplitudes of the peaks for $\delta = 0$ and π is 1/6.

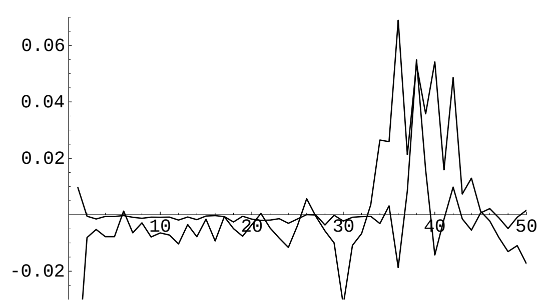

Figure 3. Ramanujan-Fourier spectrum for the cosine function of period $n_0 = 18$, with sample lengths $t = 100$ and 500. One clearly observes the compression of the lines when t increases.

Ramanujan-Fourier transform of a cosine function

Let us consider now the Ramanujan signal processing of a periodic (cosine) function of period n_0

$$a(n) = a_0 \cos(2\pi \frac{n}{n_0} + \delta).$$

The Ramanujan-Fourier coefficients read

$$a_q = \lim_{t\to\infty} \frac{a_0 \exp(i\delta)}{2\phi(q)t} \sum_p{}' \sum_{n=1}^{t} \exp[2i\pi n(\frac{p}{q} + \frac{1}{n_0})] + c.c.$$

in which the order of summations is reversed and "c.c." stands for the complex conjugate. Assume first that t is a multiple of the period n_0. Then the n-th summation is zero unless $k = \frac{p}{q} + \frac{1}{n_0}$ is a positive integer. For instance, a_q can be non-zero if $q = n_0$ under the conditions that n_0 divides $p+1$ and $k = 1$, i. e. $p = n_0 - 1$ (otherwise $p > q$, which is outside the range of summation of the p sum). One then gets

$$a_{n_0} = \lim_{t\to\infty} \frac{a_0 \exp(i\delta)}{2\phi(n_0)t} \sum_{n=1}^{t} \exp(2i\pi \frac{n}{n_0}) + c.c.$$

The n-th summation equals zero unless n_0 divides n, otherwise it equals to t. As a result, the amplitude of the n_0-th line reflects the amplitude of the periodic signal as

$$a_{n_0} = \frac{a_0}{\phi(n_0)} \cos(\delta).$$

In general, t is not a multiple of n_0 so that there exists an extra contribution to the amplitude, of order of magnitude $O(\frac{n_0}{t})$. As long as the period n_0 is much smaller than the length t of the sample, i. e. $n_0 \ll t$, one observes a single line at n_0; otherwise bursts of non-zero amplitudes emerge in the vicinity of the lines kn_0 — see Figs. 1–3.

Thus, there are two significant differences when compared to a period analysis by the standard discrete Fourier transform. First, the amplitude of the line at the period n_0 is scaled by a factor of $\phi(n_0)$. Second, the Ramanujan-Fourier analysis is sensitive to the delay δ. The latter feature may, at first sight, seem as a drawback since some period of the signal to be analyzed may be hidden by the dephasing effect. One method to circumvent this difficulty is to average the spectra corresponding to several shifted samples of the signal.

Ramanujan-Fourier transform of a period modulated cosine function

Let us now apply the approach to a period modulated cosine function. We intentionally select a period modulation with a large index (equal to 1). The selected modulation is

$$n_0 = n_0[1 + \sin(2\pi \frac{n}{n_1})],$$

with $n_0 = 10$ and $n_1 = 14$. The sample length is $t = 2000$. Due to a high modulation index, the FFT analysis (shown in Fig. 4) does not easily allow to recover the constituent integer periods 10 and 14. In contrast, the Ramanujan sum analysis is very powerful in this context. From Fig. 5 one clearly identifies (positive) large amplitudes at the periods 10, 12, 2×12 and $\mathrm{LCM}(10, 12) = 70$ (LCM being the least common multiple). Thus, for an input signal of the period n_0 and period modulation n_1, the FFT exhausts all

lines at $ln_0 + mn_1$ (l and m integers), eventually leading to a continuous spectrum in the limit of incommensurate periods n_0 and n_1. In contrast, the Ramanujan-Fourier transform is straightforward in identifying the input modulation.

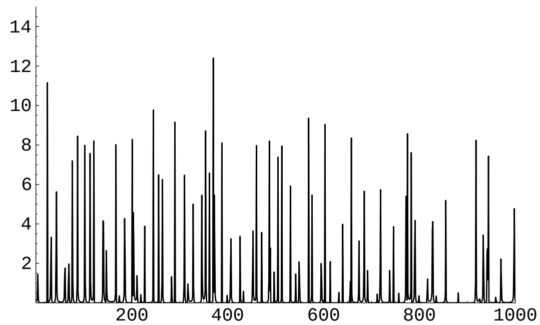

Figure 4. FFT spectrum of a period modulated cosine: $n_0 = 10$ and $n_1 = 14$.

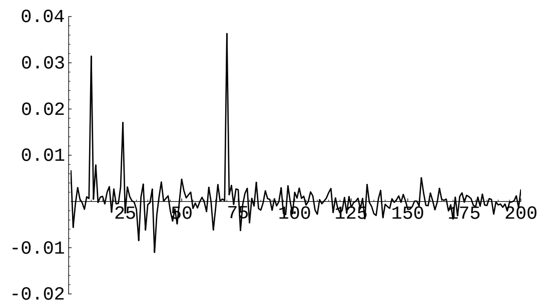

Figure 5. Ramanujan-Fourier transform of a period modulated cosine function: $n_0 = 10$ and $n_1 = 14$.

3. Ramanujan sums analysis of some complex systems

As a nice illustration of the above-outlined properties the Ramanujan sum analysis, we shall analyze a couple of complex time series taken form the stock market and solar activity.

The Dow Jones index of the stock market

The first time series deals with the Dow Jones index and has been downloaded from http : // www.optiontradingtips.com/resources/historical - data/dow-jones30.html. Fig. 6 depicts the evolution of Dow Jones 30 Industrials stock price over about 13 years. The power spectral density of the prices (Fig. 8) approximately follows a $1/f^2$ law versus the Fourier frequency $f = n^{-1}$, compatible with a Brownian-motion-based model [13, 14]. The Ramanujan-Fourier analysis shown in Fig. 7 yields a more detailed structure with many (positive or negative) peaks centered at well-identified frequencies.

There exists a sensitivity of the amplitude of the peaks (not shown) on the number t of data, but the position of the peaks, as well as their statistics, is not dependent on t. Since both spectra in Figs. 6 and 7 are given in a logarithmic time-scale, it follows that the Ramanujan sum analysis provides a clear advantage over the standard Fourier analysis in offering a rich and structured signature. Here, we shall not delve any further into the origin of this structure, which will be a topic of a separate paper.

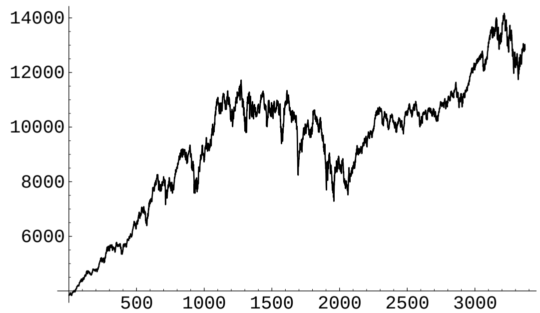

Figure 6. The Dow Jones 30 Industrials from 03.01.1995 to 30.05.2008.

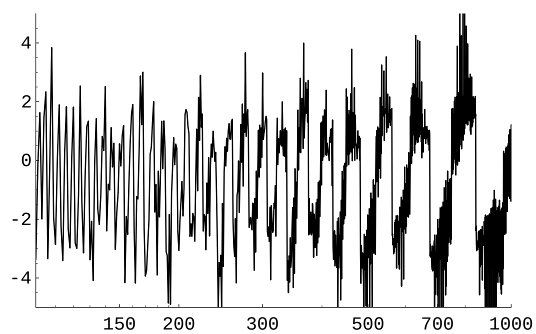

Figure 7. The Ramanujan sums analysis of the Dow Jones 30 Industrials. Periods in the range $n = 100$ to $n = 1000$ were selected. One clearly sees (positive and negative) peaks centered about non-equally-spaced periods.

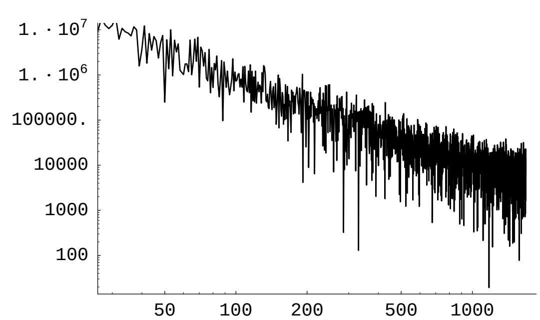

Figure 8. The FFT analysis of the Dow Jones 30 Industrials in a log-log scale. The standard $1/f^2$ dependence of the power spectrum, which is characteristic of a Brownian-like motion, is clearly visible.

The coronal index of the solar activity

The second time series has been picked up from
http : // www.ngdc.noaa.gov/stp/SOLAR/ftpsolarcorona.html#index.
It represents the Green Line (FeXIV 530.3 nm) Coronal Index of solar activity from 1939 to 2008. One easily recognizes from Figs. 9 and 10 that the coronal index is approximately periodic, with a period about 10 years. The whole FFT spectrum shown in Fig 11 exhibits a $1/f$ dependence characteristic of many physical, biological, arithmetical [7, 5, 15] and other complex systems [15].

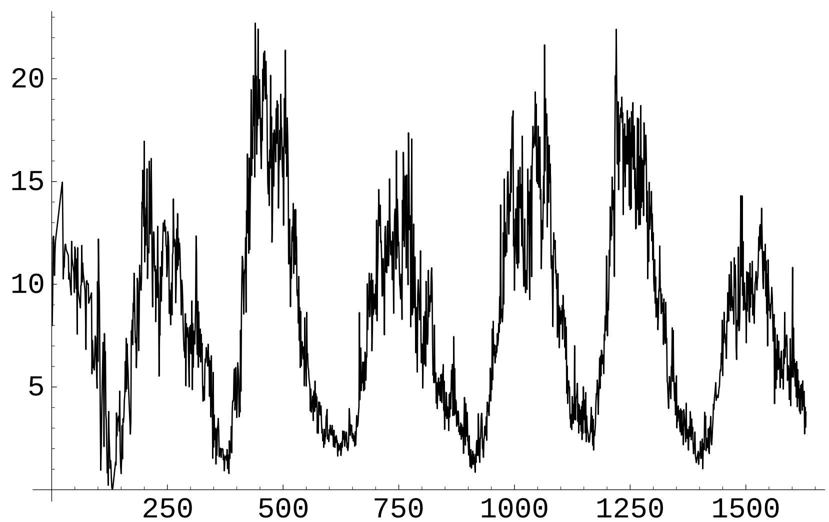

Figure 9. The temporal variation of the Coronal Index; a 10-year period is clearly visible.

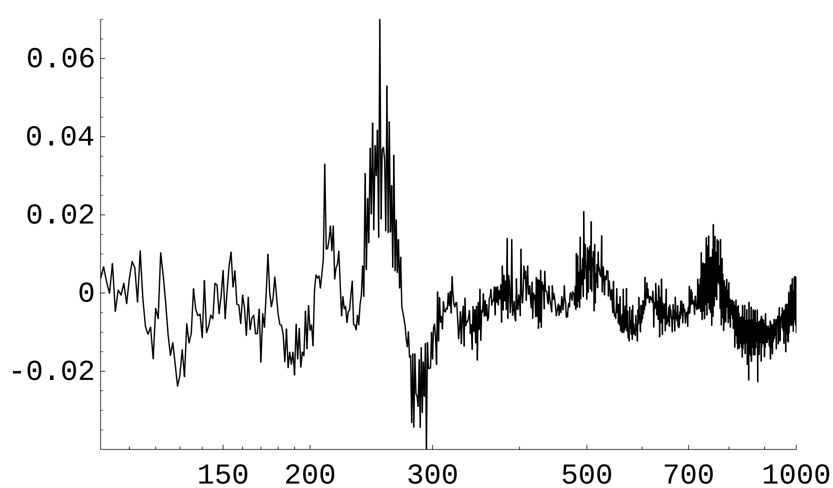

Figure 10. The RFT of the Coronal Index. The 10-year period is clearly identified; other longer periods of a smaller amplitude are observed as well, besides the harmonics pn_0 (p integer).

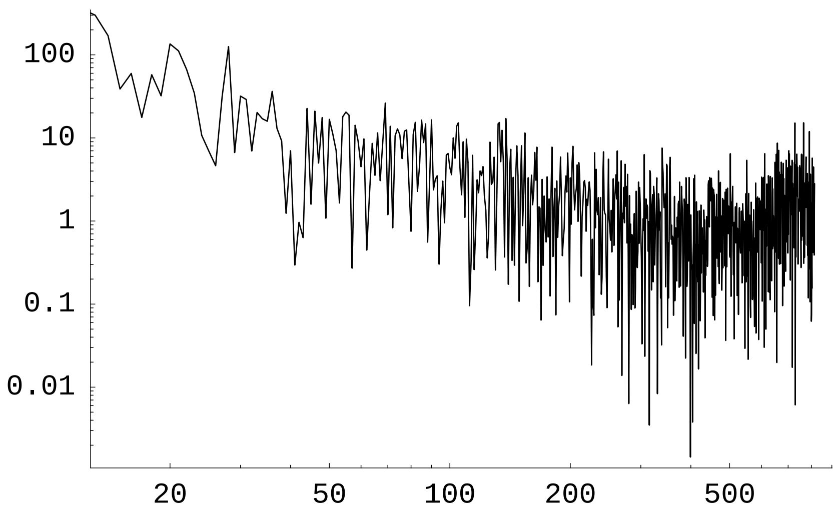

Figure 11. The FFT of the Coronal Index. One observes a $1/f$ dependence of the power spectrum; the 10-year period is quite hard to identify.

Perspectives and Conclusion

It is a widely shared belief that $1/f^\alpha$ noises are so random that non-statistical models of them are currently out of reach. A counterexample to this belief can be found in [7], in which an arithmetical approach to $1/f$ noise was suggested. In the present paper, we offer another perspective by analyzing the data from an arithmetical magnifying glass built on Ramanujan sums. The Ramanujan-Fourier transform is able to extract quasi-periodic features which are characteristic of number theoretical functions [2]-[5], as well as fine periodic features that the standard Fourier transform may hide. A Ramanujan sums analysis is a multi-scale prism with scales related to each other by the properties of irreducible fractions. It is particularly well-suited for analyzing rich time series showing a $1/f^\alpha$ $(0 < \alpha < 2)$ FFT dependence. We selected two specific complex systems to illustrate the power of this new method: the data from the stock market (for which the price index FFT follows a $1/f^2$-law) and those from solar cycle activity (for which the coronal index follows a $1/f$-law). A more detailed examination of the latter will be given in a separate paper.

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